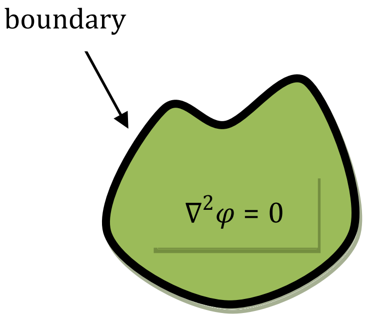
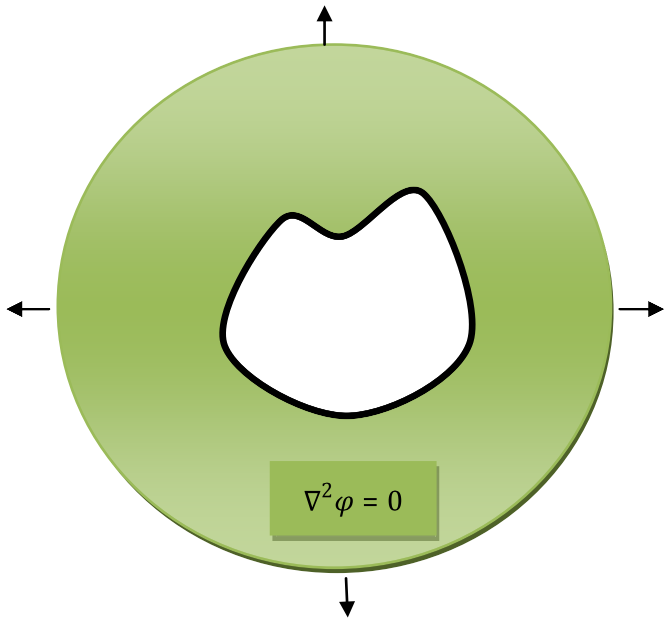
Boundary Value Problems and Boundary Conditions

A boundary value problem is a particular class of mathematical model in which a partial differential equation[1] governs a function in a defined region or regions of (usually) two-dimensional or three-dimensional space, the *problem* is to determine the (as yet unknown) function. Most simply – and most commonly – the defined region is a closed boundary. The partial differential equation model may govern the region interior to the closed boundary – an *interior problem* – or exterior to the closed boundary – an *exterior problem*.



Interior problem



Exterior Problem

In the diagrams above the interior and exterior problems are illustrated in two dimensions for simplicity. The shaded (green) area shows the region in which the partial differential is active. Note in the exterior problem the region (in theory) extends to infinity.

The partial differential equation – represented in the diagrams by Laplace's Equation – governs a function in the domain, represented in the diagram by φ. However the partial differential equation alone is insufficient to define a unique solution φ. In order to pinpoint a unique solution we require further conditions on φ, in a boundary value problem we usually set values for φ, its derivative or a combination of the two on the *boundary.* Note that different sections of the boundary may have different types of boundary condition.

<u>Dirichlet (or essential) boundary condition</u>

A Dirichlet or essential boundary condition in one in which the values of the function are defined on the boundary:

$$\varphi(\mathbf{p}) = f(\mathbf{p}) \quad \text{for } \mathbf{p} \epsilon S$$

Where $\mathbf{p}$ is any point on the boundary and $f(\mathbf{p})$ is known for those values of $\mathbf{p}$.

[1] Partial Differential Equations

Neumann (natural, derivative or *flux*) boundary condition

A Neumann, natural, derivative or flux boundary condition is one in which derivative information on $\varphi(\mathbf{p})$ is given on the boundary, usually this is expresses as the derivative with respect to the normal to the surface. For example

$$\frac{\partial\varphi}{\partial n}(\mathbf{p}) = f(\mathbf{p}) \quad \text{for } \mathbf{p}\epsilon S.$$

The normal may be defined as 'inward' or 'outward', but it needs to be defined consistently.

Robin (or mixed) boundary condition

A Robin or mixed boundary condition is one that is a hybrid of the Dirichlet nad the Neumann conditions. For example

$$\alpha\, \varphi(\mathbf{p}) + \beta\, \frac{\partial\varphi}{\partial n}(\mathbf{p}) = f(\mathbf{p}) \quad \text{for } \mathbf{p}\epsilon S.$$

where α and β are constant values.

Note that the Robin condition is a generalisation of the Dirichlet and Neumann conditions; if $\alpha=0$ then we have a Neumann condition, if $\beta=0$ then we have a Dirichlet condition.

But we can generalise further to make α and β functions of $\mathbf{p}$:

$$\alpha(\boldsymbol{p})\, \varphi(\mathbf{p}) + \beta(\boldsymbol{p})\, \frac{\partial\varphi}{\partial n}(\mathbf{p}) = f(\mathbf{p}) \quad \text{for } \mathbf{p}\epsilon S.$$

The Solution

In a real-world problem, the partial differential equation governs a region, which is demarcated by a boundary or boundaries, although the PDE can also be solved in the two- or three-dimensional free-space. If the region governed by the PDE is confined by a boundary (an *interior* problem) then it is the information on the boundary that determines the unique solution. In the case that the PDE governs a two- or three-dimensional free space, or governs the region external to a closed boundary then the behaviour in the far-field (as well as the boundary conditions) determine the unique solution.

Although in a physical reality there must be a unique solution. The related mathematical problem could, in some cases, have insufficient information to determine a unique solution. It is also possible to have contradictory information which makes a solution impossible. Such problems are said to be *ill-posed*. In general, though, a well-posed partial differential equation with a boundary condition has a unique solution. For special kinds of problems – such as those with simple geometries – it is possible to find a solution by mathematical analysis. Many real world problems are in this form, but geometries are not normally simple and numerical methods[2] are used to find an approximate solution. Typical numerical methods are the finite element method[3] and the boundary element method[4].

[2] www.numerical-methods.com
[3] www.finite-element-method.info
[4] www.boundary-element-method.com